The Black-Scholes-Merton (BSM) Option Pricing Model

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The **Black-Scholes-Merton (BSM) option pricing model** is a mathematical formula used to determine the theoretical fair price of a European-style option. It is a cornerstone of modern financial theory.

1 Key Components and Formula

The model uses five input variables to calculate the theoretical price of an option (specifically a European call or put):

Variable	**Symbol**	**Description**
Current Stock Price	S	The current market price of the underlying asset.
Strike Price	K	The price at which the option holder can buy (call) or sell (put) the underlying asset.
Time to Expiration	T	The time (in years) until the option expires.
Risk-Free Rate	r	The theoretical return on a risk-free investment (e.g., government bond yield) over the option's life.
Volatility	σ	The annualized standard deviation of the underlying asset's returns (often estimated).

The formula for a **European Call Option** price (C) is:

$$C = S \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$$

The formula for a **European Put Option** price (P) is:

$$P = Ke^{-rT} \cdot N(-d_2) - S \cdot N(-d_1)$$

Where:

- $N(\cdot)$ is the **Cumulative Distribution Function (CDF)** of the standard normal distribution . This represents the probability that a variable with a standard normal distribution will be less than or equal to a specific value.
- e is the base of the natural logarithm (approximately 2.7183).
- d_1 and d_2 are complex terms calculated from the five input variables:

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

2 Core Assumptions

The Black-Scholes model is based on several key theoretical assumptions, which are often not perfectly met in real-world markets:

- **European Options Only:** The model is only for European options, which can only be exercised at maturity.

- **Lognormal Stock Prices:** The underlying stock price follows a **geometric Brownian motion**, meaning its returns are normally distributed, but the price itself is lognormally distributed (it cannot fall below zero).
- **Constant Volatility and Risk-Free Rate:** Both the volatility (σ) and the risk-free rate (r) are assumed to be constant and known over the life of the option.
- **No Dividends:** The original model assumes the underlying asset pays no dividends. (Modifications exist to account for continuous dividend yields).
- **Frictionless Market:** There are no transaction costs, taxes, or restrictions on borrowing/lending or short selling.
- **No Arbitrage:** There are no opportunities for risk-free profit.
- **Continuous Trading:** Trading occurs continuously.

3 Derivation of the Black-Scholes PDE

The derivation of the Black-Scholes-Merton (BSM) formula is a landmark achievement in financial mathematics, combining advanced concepts from stochastic calculus and differential equations. The core idea is to construct a **risk-free portfolio** by dynamically hedging the option with the underlying stock, which eliminates all sources of randomness. By the no-arbitrage principle, this risk-free portfolio must grow at the risk-free rate.

3.1 1. Modeling the Stock Price (Geometric Brownian Motion)

The BSM model assumes that the stock price (S) follows a **Geometric Brownian Motion (GBM)**, a type of **Stochastic Differential Equation (SDE)**:

$$\mathbf{dS_t = \mu S_t dt + \sigma S_t dW_t}$$

Where:

- dS_t is the instantaneous change in the stock price.
- μ is the expected rate of return (the drift).
- σ is the volatility (diffusion).
- dW_t is a Wiener Process (or standard Brownian motion), representing the random market shock.

This SDE implies that the stock price at time T, S_T, follows a **Lognormal Distribution**. The logarithm of the stock price, $\ln(S_T)$, is normally distributed.

3.2 2. Deriving the Black-Scholes Partial Differential Equation (PDE)

A. Applying Itô's Lemma Since the option price, $V(S,t)$, is a function of the stock price (S) and time (t), and the stock price itself follows an SDE, we must use **Itô's Lemma** (the stochastic version of the chain rule) to find the SDE for the option price dV:

$$\mathbf{dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW_t}$$

- The dt term represents the predictable, deterministic change in the option's value.
- The dW_t term represents the unpredictable, stochastic (random) change.

B. Constructing the Risk-Free Portfolio A portfolio, Π, is constructed by **short-selling one option** and **long-holding Δ shares** of the underlying stock. Δ is defined as the option's sensitivity to the stock price, or $\frac{\partial V}{\partial S}$ (also known as the option's "Delta").

$$\Pi = \Delta S - V$$

The change in the portfolio value ($d\Pi$) is:

$$d\Pi = \Delta dS - dV$$

Substituting dS and dV (from the GBM and Itô's Lemma):

$$d\Pi = \Delta(\mu S dt + \sigma S dW_t) - \left[\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t \right]$$

Since $\Delta = \frac{\partial V}{\partial S}$, the $\sigma S \Delta dW_t$ terms cancel out, eliminating the random component (dW_t):

$$d\Pi = \left(\Delta \mu S - \frac{\partial V}{\partial t} - \mu S \Delta - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

C. The No-Arbitrage Condition Because the portfolio Π is now risk-free, its return must equal the risk-free rate (r) to prevent arbitrage:

$$d\Pi = r\Pi dt = r(\Delta S - V)dt$$

$$d\Pi = rS\Delta dt - rV dt$$

Equating the two expressions for $d\Pi$:

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = \left(rS \frac{\partial V}{\partial S} - rV \right) dt$$

Dividing by dt and rearranging gives the **Black-Scholes-Merton Partial Differential Equation (PDE)**:

$$\mathbf{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0}$$

Crucially, this PDE does **not** depend on the stock's expected return (μ), confirming that the option price is independent of investor risk preference, which is the foundation of **risk-neutral pricing**.

3.3 3. Solving the PDE (The Formula)

The final step is to solve the Black-Scholes PDE subject to a **boundary condition** (the option's payoff at expiration).

For a European Call Option, the terminal condition is:

$$V(S_T, T) = \max(S_T - K, 0)$$

The PDE can be transformed mathematically into the **Heat Equation** (a well-known equation in physics), for which a solution is already known.

The final solution, after solving the PDE and applying the boundary condition, is the BSM formula for the call price (C):

$$\mathbf{C = S \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)}$$

This solution can be interpreted as the discounted expected payoff of the option in a **risk-neutral world**.

- $S \cdot N(d_1)$ is the expected present value of receiving the stock, conditional on the option expiring in-the-money.
- $Ke^{-rT} \cdot N(d_2)$ is the expected present value of paying the strike price, conditional on the option expiring in-the-money.

4 The Option Greeks

The Option Greeks are a set of risk measures derived from the Black-Scholes-Merton model that quantify the sensitivity of the option price (or value, V) to changes in the underlying model parameters. They are essential tools for option traders to manage and hedge their portfolios.

4.1 1. Delta (Δ)

Delta measures the sensitivity of the option price to a change in the underlying asset's price (S). It represents the theoretical change in the option's value for a $1 change in the underlying stock price.

- **Formula (Call Option):** $\Delta_C = N(d_1)$
- **Formula (Put Option):** $\Delta_P = N(d_1) - 1$
- **Range:** Call Delta ranges from 0 to 1. Put Delta ranges from -1 to 0.
- **Interpretation:** Delta is often used as the hedging ratio, indicating the number of shares required to create a locally risk-neutral portfolio (Delta Hedging).

$$\Delta = \frac{\partial V}{\partial S}$$

4.2 2. Gamma (Γ)

Gamma measures the rate of change of Delta with respect to changes in the underlying stock price. It is the second derivative of the option price with respect to the stock price. High Gamma means Delta is very sensitive to stock price movement.

- **Formula:** $\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}}$
- **Range:** Gamma is always positive for long option positions (calls and puts).
- **Interpretation:** Gamma quantifies the risk that Delta changes significantly, requiring frequent portfolio rebalancing.

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

4.3 3. Vega ($\mathcal{V}$)

Vega (sometimes denoted by the Greek letter κ or ν) measures the sensitivity of the option price to a change in the underlying asset's volatility (σ).

- **Formula:** $\mathcal{V} = S\sqrt{T}N'(d_1)$
- **Range:** Vega is always positive for long option positions, meaning option prices increase when volatility rises.
- **Interpretation:** Vega is crucial for managing risk associated with shifts in market expectations of future volatility.

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

4.4 4. Theta (Θ)

Theta measures the sensitivity of the option price to the passage of time (T). It quantifies the rate of time decay.

- **Formula (Call Option):** $\Theta_C = -\frac{S\sigma N'(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2)$
- **Formula (Put Option):** $\Theta_P = -\frac{S\sigma N'(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$
- **Range:** Theta is typically negative for long option positions, meaning the option value decreases as time passes (time decay).

- **Interpretation:** Theta is the cost of holding an option; it is the amount of value an option loses each day, assuming all other factors remain constant.

$$\Theta = -\frac{\partial V}{\partial T}$$

4.5 5. Rho (ρ)

Rho measures the sensitivity of the option price to a change in the risk-free interest rate (r).

- **Formula (Call Option):** $\rho_C = KTe^{-rT}N(d_2)$
- **Formula (Put Option):** $\rho_P = -KTe^{-rT}N(-d_2)$
- **Interpretation:** Rho is generally small but becomes more important for long-term options, as the impact of compounding interest rates is magnified over longer periods.

$$\rho = \frac{\partial V}{\partial r}$$