From Implied Volatility Smile to Risk–Neutral Density: An Explicit Derivation Using SVI

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1 Introduction

European option prices uniquely determine the risk–neutral probability distribution of the underlying asset at maturity. This fundamental result, due to Breeden and Litzenberger [1], implies that the implied volatility smile observed in the market encodes the full marginal distribution of the underlying under the pricing measure.

In this note we derive explicitly the risk–neutral density implied by a given implied volatility smile, focusing on the widely used Stochastic Volatility Inspired (SVI) parameterization [2].

2 Risk–Neutral Density from Option Prices

Let S_T denote the underlying price at maturity T. Under the risk–neutral measure $\mathbb{Q}$, the price of a European call option with strike K is

$$C(K,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right]. \tag{1}$$

Breeden and Litzenberger [1] showed that, provided sufficient smoothness,

$$f^{\mathbb{Q}}_{S_T}(K) = e^{rT}\frac{\partial^2 C(K,T)}{\partial K^2}, \tag{2}$$

where $f^{\mathbb{Q}}_{S_T}$ is the risk–neutral density of S_T.

3 Forward Measure and Log–Moneyness

Define the forward price

$$F = S_0 e^{(r-q)T}, \tag{3}$$

and log–moneyness

$$k = \log\left(\frac{K}{F}\right). \tag{4}$$

Let the total implied variance be

$$w(k) = \sigma^2_{\text{impl}}(K,T)\,T. \tag{5}$$

In forward measure, the Black–Scholes call price is

$$C^F(k) = F\left[N(d_+) - e^k N(d_-)\right], \tag{6}$$

with

$$d_\pm(k) = \frac{-k}{\sqrt{w(k)}} \pm \frac{1}{2}\sqrt{w(k)}. \tag{7}$$

The density in log–moneyness coordinates is

$$f_k(k) = \frac{\partial^2 C^F(k)}{\partial k^2}, \tag{8}$$

and the density of S_T follows from the Jacobian:

$$f_{S_T}^{\mathbb{Q}}(K) = \frac{1}{K}\, f_k(\log(K/F)). \tag{9}$$

4 General Density Formula from a Smile

For any twice differentiable total variance function $w(k)$, the log–density admits the closed–form expression [3]

$$f_k(k) = \frac{1}{\sqrt{2\pi w(k)}} \exp\left(-\frac{d_-^2(k)}{2}\right)\left[1 - \frac{kw'(k)}{2w(k)} + \frac{(w'(k))^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) - \frac{w''(k)}{2}\right], \tag{10}$$

where primes denote derivatives with respect to k.

Positivity of the bracketed term is equivalent to absence of butterfly arbitrage.

5 SVI Parameterization

The raw SVI parameterization of total implied variance is given by [2]

$$w(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right), \tag{11}$$

with parameters

$$a \in \mathbb{R}, \quad b > 0, \quad |\rho| < 1, \quad \sigma > 0, \quad m \in \mathbb{R}.$$

Define

$$\Delta = k - m, \qquad D = \sqrt{\Delta^2 + \sigma^2}.$$

The first and second derivatives are

$$w'(k) = b\left(\rho + \frac{\Delta}{D}\right), \tag{12}$$

$$w''(k) = b\,\frac{\sigma^2}{D^3}. \tag{13}$$

6 Explicit Risk–Neutral Density

Substituting the SVI derivatives into the general density formula yields

$$f_k(k) = \frac{1}{\sqrt{2\pi w(k)}} \exp\left(-\frac{1}{2} \left(\frac{-k}{\sqrt{w(k)}} - \frac{1}{2}\sqrt{w(k)} \right)^2 \right) G(k), \tag{14}$$

where

$$\begin{aligned} G(k) = 1 - \frac{k}{2w(k)}\, b\left(\rho + \frac{\Delta}{D}\right) \\ + \frac{b^2}{4}\left(\rho + \frac{\Delta}{D}\right)^2 \left(\frac{1}{w(k)} + \frac{1}{4}\right) - \frac{b}{2}\frac{\sigma^2}{D^3}. \end{aligned} \tag{15}$$

Finally, the risk–neutral density of the underlying price is

$$f^{\mathbb{Q}}_{S_T}(K) = \frac{1}{K}\, f_k(\log(K/F)) \,. \tag{16}$$

7 Conclusion

The implied volatility smile, through its total variance function and derivatives, uniquely determines the risk–neutral probability distribution of the underlying. In the case of SVI, this relationship is fully explicit and model–free: the smile itself *is* the marginal distribution.

References

[1] D. T. Breeden and R. H. Litzenberger, "Prices of state-contingent claims implicit in option prices," *Journal of Business*, vol. 51, no. 4, pp. 621–651, 1978.

[2] J. Gatheral, "A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives," *Proceedings of the Global Derivatives & Risk Management Conference*, 2004.

[3] J. Gatheral, *The Volatility Surface: A Practitioner's Guide*, Wiley Finance, 2011.