Why Option Prices Can Be Written as Fourier Integrals

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1. Starting From the Risk–Neutral Expectation

For a European call option maturing at T with strike K, the price is

$$C(K) = e^{-rT}\, E[(S_T - K)^+].$$

Let us write the log-price as

$$X_T = \log S_T.$$

Then

$$S_T = e^{X_T}, \quad (S_T - K)^+ = (e^{X_T} - K)^+.$$

Expanding the expectation using the risk-neutral density $f_{X_T}(x)$:

$$C(K) = e^{-rT} \int_{\log K}^{\infty} (e^x - K)\, f_{X_T}(x)\, dx.$$

This shows that if we know the density of X_T, then pricing is simply an integral. The *Fourier approach* replaces the density by its Fourier-transform representation.

2. Why the Density Can Be Written as an Inverse Fourier Integral

Let $\phi_{X_T}(u)$ be the characteristic function of X_T:

$$\phi_{X_T}(u) = E[e^{iuX_T}].$$

The Fourier inversion theorem states that if X_T has a density $f_{X_T}(x)$, then

$$f_{X_T}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{X_T}(u) e^{-iux}\, du.$$

This is a fundamental harmonic–analysis result: the characteristic function uniquely determines the probability density, and the density can be recovered by taking the inverse Fourier transform of the characteristic function.

3. Substituting the Fourier Representation of the Density

Insert the inverse Fourier integral for the density into the option pricing formula. Start from

$$C(K) = e^{-rT} \int_{\log K}^{\infty} (e^x - K)\, f_{X_T}(x)\, dx.$$

Replace $f_{X_T}(x)$:

$$C(K) = e^{-rT} \int_{\log K}^{\infty} (e^x - K) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{X_T}(u) e^{-iux}\, du \right] dx.$$

Interchange integrals (justified under mild integrability conditions):

$$C(K) = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} \phi_{X_T}(u) \left[\int_{\log K}^{\infty} (e^x - K) e^{-iux} dx \right] du.$$

Now the option price appears as a *Fourier transform* of a known kernel times the characteristic function.

This is the key step:

Option prices can be expressed as Fourier integrals because the density itself is the inverse Fou

4. Cleaning the Expression

Define $k = \log K$. The inner integral has closed form for all u. Thus the entire pricing formula becomes

$$C(K) = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi(u)\, du,$$

where $\psi(u)$ is an explicit expression built from ϕ_{X_T}. This is exactly a Fourier transform.

5. Why FFT Is Useful

The expression above has the generic structure

$$\text{Price}(K) = \int e^{-iuk}\, G(u)\, du,$$

which is a Fourier transform of the function $G(u)$. Once written in this form:

- the integral becomes a discrete Fourier transform (DFT),
- which can be computed by the fast Fourier transform (FFT),

- allowing hundreds of strikes to be priced in one shot.

Thus the FFT is not the model itself: it is simply the computational engine used because the pricing formula reduces to a Fourier transform.

6. Summary

1. Option price = e